

# OPTIMAL TIME ADVANCE IN TERMINAL AREA ARRIVALS: THROUGHPUT VS. FUEL SAVINGS

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## Abstract

The current operational practice in scheduling air traffic arriving at an airport is to adjust flight schedules by *delay*, i.e. a postponement of an aircraft's arrival at a scheduled location, to manage safely the FAA-mandated separation constraints between aircraft. To meet the observed and forecast growth in traffic demand, however, the practice of *time advance* (speeding up an aircraft toward a scheduled location) is envisioned for future operations as a practice additional to delay. Time advance has two potential advantages. The first is the capability to minimize, or at least reduce, the excess *separation* (the distances between pairs of aircraft immediately in-trail) and thereby to increase the throughput of the arriving traffic. The second is to reduce the total traffic delay when the traffic sample is below saturation density. A cost associated with time advance is the fuel expenditure required by an aircraft to speed up. We present an optimal control model of air traffic arriving in a terminal area and solve it using the Pontryagin Maximum Principle. The admissible controls allow time advance, as well as delay, some of the way. The cost function reflects the trade-off between minimizing two competing objectives: excess separation (negatively correlated with throughput) and fuel burn. A number of instances are solved using three different methods, to demonstrate consistency of solutions.

## Nomenclature

a/c	aircraft
$I$	the number of flights
$i$	the index of the a/c on the $i$ -th flight
$\tau(i)$	the aircraft type of the a/c on the $i$ -th flight
$s_i$	the arc length coordinate of the $i$ -th a/c
$\mathbf{s}$	the vector $(s_1, s_2, \dots, s_I)$
$\underline{v}_i$	the minimal cruising speed of the $i$ -th a/c
$\overline{v}_i$	the maximal cruising speed of the $i$ -th a/c
$v_i$	the actual or assigned speed of the $i$ -th a/c
$\mathbf{v}$	the vector $(v_1, v_2, \dots, v_I)$ of control speeds
$B_{\tau(i)}(v)$	fuel burn for an a/c of type $\tau(i)$ at speed $v$
$v_i^{ecs}$	the "economical cruising speed" of a/c $i$ , at which the fuel burn is minimal
$T$	exit time for problem P1 (definition 2)
$\alpha, \beta$	constant parameters in $B_{\tau(i)}(v_i)$ (see (11))
$\langle, \rangle$	the Euclidean <i>inner product</i> [1, section 5.2-6]
$\ \cdot\ $	the <i>norm</i> [1] arising from $\langle, \rangle$
$\stackrel{\text{def}}{=}$	"is defined as"
$\mu_{i,i'}$	required min. separation for aircraft $i$ directly followed by aircraft $i'$
$E(\mathbf{s})$	excess separation (13)
$c$	relative importance coeff. of $B_{\tau(i)}(v_i)$ to $E(\mathbf{s})$
P1	the optimal control problem for the Delay-Time Advance flight phase (definition 2)
$L_i$	distance the $i$ -th a/c is to fly from the end of the Delay-Time Advance phase to landing

## Introduction

The current operational practice in scheduling air traffic arriving at an airport is to adjust flight schedules by *delay*, i.e. a postponement of an aircraft's arrival at a scheduled location, to manage safely the separation constraints between aircraft. This practice is most effective when the system is near saturation or capacity, leaving delay as the only control option. Emphasis is placed on delay distribution between the Center and the TRACON (see refer-

ence [2, section 2.2.5] and references therein). In this practice, *delay* is preferred to a speed-up known as *time advance*. The practice of time advance, although not in use today, is envisioned for future operations in light of the observed and forecast growth in traffic demand. The ability to practice time advance can reduce overall system delay in two types of situations: *i)* in the face of impending saturation, and *ii)* when the state of traffic alternates between saturation and under-saturation. A cost incurred by an aircraft exercising time advance is increased fuel expenditure. This calls for a study of the trade-off between time advance and fuel burn.

The focus of this paper is such a study, carried out using optimal control theory [3], and leading to a computationally efficient procedure for producing speed advisories. Air traffic arriving into a terminal airspace, with multiple runways allowed, is modeled using optimal control, with the state space constructed as in [4]; the portion of this modeling framework used in this paper is formulated in the section *A mathematical setting for optimal control models with explicit delays*.

A number of past research efforts have focused on the trade-off between delay and fuel burn [5, 6]. An approach widely used in general Air Traffic Management (ATM) research is to cast the problem as a mixed-integer (non)linear program [7–10]. Since the sets of flights, waypoints, meter fixes, and route segments are finite, it is natural to think of the operational problems intuitively as discrete. The mixed-integer programming framework corresponds directly to this intuition, hence is a convenient model for capturing the problem realistically. In the context of modeling with a view toward future operational use, however, this framework faces the challenges of *i)* a lack of qualitative insight into the behavior of optimal solutions, and *ii)* absence of proofs of convergence or of bounds on the computational cost, or of both.

One advantage of optimal control over other areas of optimization is the availability of a general theory, notably the *Bellman Optimality Principle* and the *Pontryagin Maximum Principle* [3]. This theory offers insight into the qualitative behavior of optimal solutions, often before a solution is computed, and can yield computational solution procedures that come with mathematical proofs of *cor-*

*rectness* (i.e., that a computed result is in fact a solution to the problem) and of *low computational complexity bounds* (i.e., that the computation will complete in a low-degree polynomial time) [11, 12]. Such proofs are a highly desired feature for an algorithm that drives an automated tool for supporting safety-critical real-time ATM operations.

The solutions found in this paper for problem P1 (see definition 2) make the following three main contributions:

- a demonstration that the model is scalable;
- an efficient way (eligible for real-time use) to compute optimal speed advisories accurately;
- insight into the relation between fuel burn, incurred by time advance, and throughput, characterized by excess separation in the *Delay-Time Advance* phase of the flight.

## Background

In general, the arrival of air traffic can be divided into the following phases.

- *Delay-Time Advance phase:* All aircraft are still able to exercise delay or time advance.
- *Mixed phase:* At least one aircraft can no longer exercise delay or time advance, and at least one other aircraft still can.
- *Separation Minimization phase:* Aircraft can no longer exercise delay or time advance.

The focus of this paper is on the Delay-Time Advance phase. This phase is modeled, below, as an optimal control problem which includes a *running cost* term reflecting fuel burn. Two assumptions central to the model are as follows:

**Assumption 1** In the Delay-Time Advance phase, an optimal trajectory does not touch the boundary of the set of conflicting states (i.e., states in which at least one pair of aircraft violates the separation constraint).

**Assumption 2** The aircraft in question are flying in zero-wind conditions.

One may question whether the mathematical model developed here with the use of the simplifying Assumption 2 is readily generalized to include reasonable uncertainties, such as wind. The role of Assumption 2 in this study is as follows. Since an aircraft's navigation system will generally keep the aircraft on the prescribed path, the main effect of a wind field is from its component tangential to the path. The effect of uncertainties, such as wind or error in control execution, on the model is an introduction of error terms into the endpoints of the aircraft's speed range:

$$\begin{aligned} & (\text{min. speed}) + (\text{error due to uncertainty}) \\ & \leq (\text{the aircraft's ground speed}) \\ & \leq (\text{max. speed}) + (\text{error due to uncertainty}) \end{aligned}$$

Trajectories allowed to touch conflict boundaries before reaching minimal separation are beyond the scope of this paper. Optimal control models corresponding to the Mixed phase and the Separation Minimization phase are given, respectively, in appendices I and II.

All the operational decisions concerning the use of delay or time advance are made in the flight phase corresponding to optimal control problem P1, formulated below, with a weighted objective function of the form

$$\begin{aligned} & (\text{coeff. of relative utility})(\text{collective fuel burn}) \\ & + (\text{excess separation}) \end{aligned}$$

A solution to the problem with a total of  $I$  aircraft is a time-dependent vector-valued function

$$\mathbf{v}(t) = (v_1(t), \dots, v_I(t)), \quad (1)$$

where  $v_i(t)$  is the speed dictated to aircraft  $i$  at time  $t$ .

**Remark 1** Operationally, it is desirable that the control strategy  $\mathbf{v}(t)$  be piecewise constant: in this case, every time interval of constancy can be regarded as a speed advisory, and  $\mathbf{v}(t)$  is then a sequence of speed advisories for the entire set of aircraft.

A general solution is found below as a root of an exactly known polynomial. For the cost functions considered, the mathematical model indicates piecewise constant control strategies  $\mathbf{v}(t)$ . Two by-products of the solution are as follows:

- a functional interdependence between the speed advisories and the above coefficient of relative utility, and
- a way to compute—once the solutions for other flight phases are found—the desired change (a delay if positive, a time advance if negative) to the scheduled arrival time (STA); see the section *Obtaining delays and time advances from a solution*.

Solutions to a number of problem P1 instances are presented in a later section.

## A mathematical setting for optimal control models with explicit delays

### The state space

Consider a collection of  $P$  paths in  $\mathbf{R}^3$ , indexed by the set  $\mathcal{P} = \{1, \dots, P\}$ . Path  $p \in \mathcal{P}$  is assumed to admit a continuous and piecewise continuously differentiable one-to-one parameterization by its arc length  $s_p$ . Thus, if  $\vec{x}_p$  is the mentioned spatial parameterization of path  $p$ , then

$$\vec{x}_p(s_p) \in \mathbf{R}^3, \quad -\infty \leq s_p \leq 0 \quad \text{for all } p \in \mathcal{P} \quad (2)$$

**Definition 1** The point at  $s_p = 0$  in parameterization (2) will be called the distal point of path  $p$ .

A pair of paths can either be disjoint, or intersect at a point (Figure 1A), or overlap (Figure 1B). The distal points in Figure 1 are shown as black dots; the parameters  $\alpha$  decrease, in the direction of aircraft motion, toward the distal point. The portions of the paths near the crossing (overlap) play a role in the geometry of the conflicting states, as will be shown in Figure 2.

Suppose each aircraft in a set  $\mathcal{A} = \{1, 2, \dots, I\}$  is assigned one of the paths in  $\mathcal{P}$ . To specify that a path  $p \in \mathcal{P}$  has aircraft  $i$  assigned to it, we will write  $p(i)$  instead of  $p$ . The following definition of a state space follows the approach first introduced in [4]. Since the parameterization mapping  $\vec{x}_p$  is one-to-one and onto, the spatial position of aircraft  $i$  is completely determined by specifying its distance,

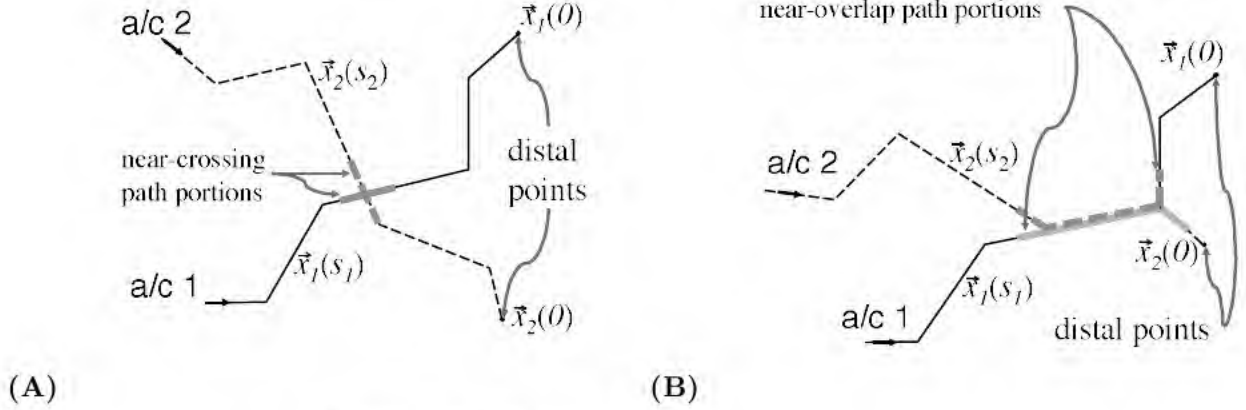


Figure 1: Path pairs.

measured along the path, from the distal point (definition 2). Let  $-s_i$  be aircraft  $i$ 's distance, measured along the path, from the distal point. This definition implies that if aircraft  $i$  is on path  $p(i)$ , then  $s_i = s_{p(i)}$ . Thus, if each aircraft has already entered its path, then by specifying all the  $s_i$ 's (each  $s_i$  increases with time), one specifies each aircraft's spatial position. Accordingly, in the control model constructed below, the vector

$$\mathbf{s} = (s_i)_{i \in \mathcal{A}}$$

is a *state* vector, hence is regarded as a function of time:

$$\mathbf{s} = \mathbf{s}(t) = (s_i(t))_{i \in \mathcal{A}}$$

It follows that the state space of the problem is the entire  $I$ -dimensional Euclidean space

$$\mathbf{S} = \mathbf{R}^I$$

Different portions of this space will be used below in the optimal control models that correspond to different flight phases.

The rest of this section consists of some specifications of the above machinery to the models below. The distal point (definition 2) will be assumed to be a runway. Two different paths may terminate with two different runways. The following two operational restrictions will be reflected:

1. On reaching a certain distance from the runway, aircraft  $i$  can no longer exercise delay or

time advance. This distance will be denoted  $L_i$ .

2. On reaching a certain distance from the runway, aircraft  $i$  must assume and keep the landing speed. This distance will be denoted  $l_i$

These restrictions are related to the state variables as follows: aircraft  $i$  can no longer exercise delay or time advance if  $s_i > L_i$ , and must move at landing speed if  $s_i > l_i$ . The latter state space construct is used below to formulate a class of optimal control problems.

### Separation constraints and landing sequence

Denoting the minimal separation distance for an ordered pair  $(i, i')$  of aircraft by  $\rho_{i, i'}$ , and letting  $\|\cdot\|$  denote the Euclidean norm in  $\mathbf{R}^3$ , we obtain the *separation constraints*

$$\left. \begin{aligned} \|\vec{x}_{p(i)}(s_i(t)) - \vec{x}_{p(i')}(s_{i'}(t))\| &\geq \rho_{i, i'} \\ \text{for all } i, i' \in \mathcal{A} \text{ and all meaningful } t \end{aligned} \right\} \quad (3)$$

If the paths of a pair  $(i, i')$  of aircraft join once and coincide henceforth, the set of states where these two aircraft are in conflict can be approximated from the outside by a polyhedral region, henceforth



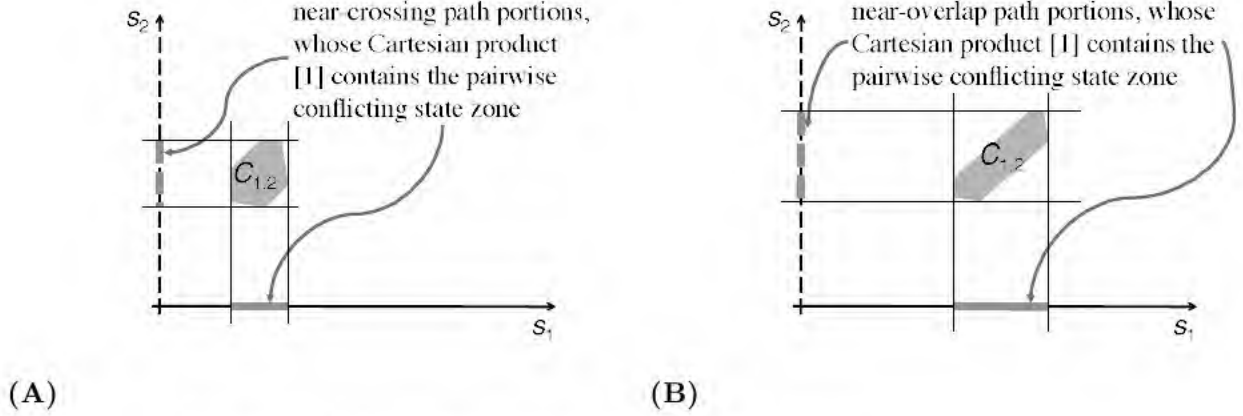


Figure 2: Approximate pairwise conflict zones.

called a *pairwise conflicting state zone* and denoted  $C_{i,i'}$ , whose dimensions are determined by the required minimal separation distances, and whose diagonal extent is commensurate with the length of the common portion of the two paths. For example, the polygonally approximated conflict zones corresponding to the path pairs shown in Figure 1 are shown in Figure 2. The resulting form for the separation constraints is

$$\mathbf{s} \notin \bigcup_{i,i' \in \mathcal{A}} C_{i,i'} \quad (4)$$

**Remark 2** *The separation distance required for a pair of aircraft generally depends on their wake classes and on the order in which the two aircraft consecutively enter a common path. This implies the possibility of the asymmetry  $\rho_{i,i'} \neq \rho_{i',i}$ .*

### Landing sequence and the minimal separation line (MSL)

In this paper, the landing sequence is assumed predetermined. (While this assumption can be justified operationally today, it may lose basis in future ATM operations and research. The computational implications of abandoning the assumption are summarized briefly in the section *Discussion*.) A given landing sequence can be described as a permutation  $\pi$  on the set  $\mathcal{A} = \{1, 2, \dots, F\}$  such that, at the time when the first aircraft in the set lands and all the

others still observe separation, the state vector  $\mathbf{s}$  satisfies the inequalities

$$s_{\pi(1)} < s_{\pi(2)} < \dots < s_{\pi(I)} \quad (5)$$

Thus, the permutation  $\pi$  “sorts the aircraft in the order of reaching their distal points”, which in this case are the runway or runways.

Given a landing sequence  $\pi$ , a state  $\mathbf{s}$  with minimal separation between every consecutive pair of aircraft satisfies the following, stronger, version of the inequalities (5):

$$s_{\pi(i)} + \rho_{\pi(i), \pi(i+1)} = s_{\pi(i+1)}, \quad i = 1, 2, \dots, I-1 \quad (6)$$

The system (6) of  $(I-1)$  equations in the  $I$ -dimensional space  $\mathbf{R}^I$  determines a line, called the *minimal separation line (MSL)*.

### Control law

Ignoring inertia, the speed  $v_i$  of aircraft  $i$  is assumed capable of being arbitrarily set to any value in the operationally admissible range. This leads to the following control law:

$$\dot{\mathbf{s}} = \mathbf{v}, \quad (7)$$

where

$$\mathbf{v} = (v_1, \dots, v_I)$$

The operationally admissible speed ranges will be specified for each of the control problems formulated below.

# Delay-time advance phase with fuel cost: optimal control problem P1

## Problem formulation

Operationally, flight  $i$  is assigned a tentative *scheduled time of arrival (STA)*, denoted  $STA_i$ , which marks the end of the flight's Delay-Time Advance phase. If this assignment were never subject to change, the appropriate condition for flight  $i$  to exit this flight phase would be

$$s_i(STA_i) = -L_i$$

However, STAs are updated in operations, typically by requiring postponement of arrival by a time duration called *delay*, necessitated when the level of traffic demand reaches the throughput capacity and, without rescheduling, will lead to separation violations. Allowing not only to postpone phase exit, but also to expedite it (i.e., allowing a *time advance*), carries a potential operational gain: during periods when traffic demand is below saturation, time advance can ultimately allow aircraft to be landed earlier than originally planned. Thus, with a corresponding adjustment and optimization in runway scheduling, a practice of time advance can help increase throughput.

In the present model, delay and time advance are represented by introducing for each aircraft  $i$  a variable  $\delta_i$  whose value is to be determined as part of solving the optimization problem. This results in the more realistic condition

$$s_i(STA_i + \delta_i) = -L_i$$

for flight  $i$  to exit the Delay-Time Advance phase.

In this paper, sufficiently early knowledge of an aircraft's whereabouts (starting at a time, denoted by  $t^0$ ) is assumed. This gives the following initial condition:

$$\mathbf{s}(t^0) = \mathbf{s}^0 \in \mathbf{S}^0, \quad t^0 \leq \min_i (STA_i + \delta_i) \quad (8)$$

The quantities  $\delta_i$  are unknown at the outset and can be found, once solutions for all three flight phases are obtained, using formula (22).

The Delay-Time Advance phase of an aircraft  $i$ 's flight lasts until  $s_i$  reaches the value  $-L_i$ . Accordingly, denoting by  $T$  the exit time for the problem, the target set is defined for this problem as

$$\mathbf{S}^{T,P1} = \left\{ \mathbf{s} : \begin{array}{l} s_i \leq -L_i \text{ for all } i, \text{ with} \\ \text{equality for at least one } i. \end{array} \right\} \quad (9)$$

For a case of 2 a/c, Figure 3 schematically shows the state space for this problem as the quadrant  $\{(s_1, s_2) : s_1 \leq -L_1, s_2 \leq -L_2\}$  shaded gray, with an example of an initial state (8).

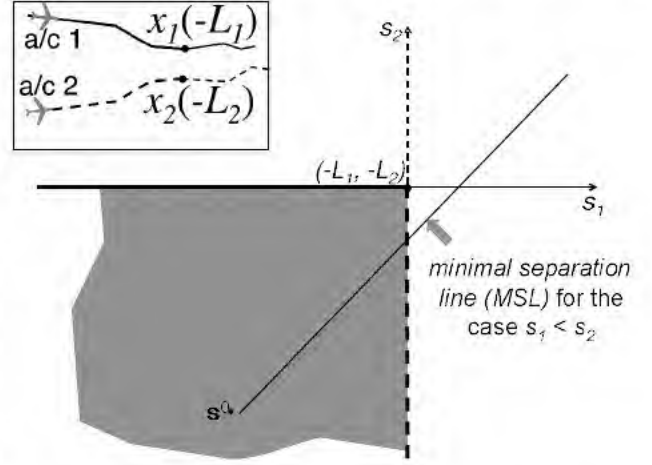


Figure 3: A 2-a/c state space for P1.

The speed range for aircraft  $i$  is the full cruise speed range  $[\underline{v}_i, \bar{v}_i]$ , which contains a “economical cruising speed”  $v_i^{ecs}$ , closer to  $\underline{v}_i$  than to  $\bar{v}_i$ , that minimizes the instantaneous rate (11) of fuel burn. For future reference, we write the constraints on the speeds:

$$\underline{v}_i \leq v_i \leq \bar{v}_i \quad i = 1, 2, \dots, I \quad (10)$$

The running cost of for aircraft  $s_i$  with  $s_i < -L_i$  is the instantaneous rate of fuel burn,  $B_{\tau(i)}$ , expressed as a known function of the aircraft's speed  $v_i$ . Based on [13],  $B_{\tau(i)}$  is a convex function on  $[\underline{v}_i, \bar{v}_i]$  with a minimum at some “economical cruising speed”  $v_i^{ecs}$ .

For simplicity, all  $v_i^{ecs}$ 's will be assumed to have the same value, henceforth denoted  $\beta$ . All theoretical results of this paper, however, easily generalize to the case when the  $v_i^{ecs}$ 's are not necessarily equal.

Throughout this paper, the functional form

$$B_{\tau(i)}(v_i) = 1 + \alpha(v_i - \beta)^2, \quad (11)$$

with  $\beta = v_i^{ccs}$ , is used for the instantaneous rate of fuel burn for each  $i$ , to mimic the data shown in reference [13, figure 1]. Although symmetric about  $\beta$ , the form (2) exhibits qualitative behavior that resembles the asymmetric form in [13, figure 1] sufficiently for our first-step investigation.

The total running cost is, therefore,

$$c \sum_{i=1}^I B_{\tau(i)}(v_i), \quad (12)$$

where the coefficient  $c$  measures the relative utility of minimizing fuel burn to minimizing excess separation. The functions  $B_{\tau(i)}$  are assumed quadratic polynomials, which can be justified by interpolating  $B_{\tau(i)}$  through 3 data points.

Let  $\mathbf{s}^{MSL}$  be the intersection between the MSL (6) and the target set  $\mathbf{S}^{T,P1}$  defined in (9). To incorporate the goal of minimizing separation, the exit cost for problem P1 is defined as

$$E(\mathbf{s}) = \frac{1}{2} \|\mathbf{s} - \mathbf{s}^{MSL}\|^2, \quad (13)$$

where  $\|\cdot\|$  is the *norm* (reference [1, section 5.2-5]) arising from the *inner product* (reference [1, section 5.2-6]), denoted here by  $\langle \cdot, \cdot \rangle$ . Note that  $s_{\pi(I)}^{MSL} = -L_{\pi(I)}$ , and the rest of the  $s_i^{MSL}$ 's can be directly calculated from (6).

**Definition 2** *The optimal control problem given by (7), (8), (9), (10), (12), (13) will henceforth be called problem P1.*

### **Solutions for problem P1 obtained using the Pontryagin Maximum Principle**

Denote the costate variable corresponding to  $s_i$  by  $\sigma_i$  and write  $\boldsymbol{\sigma} = (\sigma_1, \dots, \sigma_I)$ . The Pontryagin function (also known as the *variational Hamiltonian*) for the problem is then

$$H = -c \sum_i B_{\tau(i)}(v_i) + \langle \mathbf{v}, \boldsymbol{\sigma} \rangle \quad (14)$$

The adjoint state [3] equations

$$\dot{\boldsymbol{\sigma}} = -\frac{\partial}{\partial \mathbf{s}} H = \mathbf{0}$$

imply that  $\boldsymbol{\sigma}$  is constant. From the constancy of  $\boldsymbol{\sigma}$ , we conclude that there exists a maximizing control

strategy,  $\mathbf{v}$ , which is constant. (The sample numerical solutions obtained using the Dynamic System Optimization Algorithm (OCP) software [14] are consistent with the latter conclusions.)

**Remark 3** *The qualitative insight gained from the above application of the Pontryagin Maximum Principle is that problem P1 has an optimal control strategy  $\mathbf{v}^*$  which is constant. This insight not only shows that  $\mathbf{v}^*$  satisfies the operational preference expressed in remark 1, but also reduces a problem in optimal control to one in static optimization,*

Such a constant control strategy can be found by a direct polynomial-time computation as follows.

The total cost of a trajectory  $\mathbf{s}(t)$  corresponding to a control strategy

$$\mathbf{v}(t) = \mathbf{v} = \text{const.} \quad (15)$$

is

$$\Phi = \int_{t=t^0}^{t=T} c \sum_i B_{\tau(i)}(v_i) dt + \frac{1}{2} \|\mathbf{s}^1 - \mathbf{s}^{MSL}\|^2$$

Recalling that

$$\|\mathbf{s}^1 - \mathbf{s}^{MSL}\|^2 = \langle \mathbf{s}^1 - \mathbf{s}^{MSL}, \mathbf{s}^1 - \mathbf{s}^{MSL} \rangle,$$

and that, in light of (15), the final state  $\mathbf{s}^1$  is given by

$$\mathbf{s}^1 = \mathbf{s}^0 + \int_{t=0}^T \mathbf{v} dt = \mathbf{s}^0 + T\mathbf{v},$$

we obtain

$$\begin{aligned} \Phi &= cT \sum_i B_{\tau(i)}(v_i) + \frac{1}{2} \|\mathbf{s}^0 - \mathbf{s}^{MSL}\|^2 \\ &+ \frac{1}{2} T^2 \|\mathbf{v}\|^2 + T \langle \mathbf{s}^0 - \mathbf{s}^{MSL}, \mathbf{v} \rangle \end{aligned}$$

Omission of constant terms from the latter expression leaves the solutions of the minimization problem ( $\min \rightarrow \Phi$ ) unchanged, and therefore the optimal control problem can be stated as

$$\begin{aligned} \min_{\mathbf{v}, T} \rightarrow \tilde{\Phi} &\stackrel{\text{def}}{=} cT \sum_i B_{\tau(i)}(v_i) \\ &+ \frac{1}{2} T^2 \|\mathbf{v}\|^2 + T \langle \mathbf{s}^0 - \mathbf{s}^{MSL}, \mathbf{v} \rangle, \end{aligned} \quad (16)$$

subject to the constraint

$$g \stackrel{\text{def}}{=} s_1^0 + T v_1 - (-L_1) = 0$$

The function  $\tilde{\Phi}$ , defined in (16), can be written

$$\tilde{\Phi} = T \left[ a_2(T) \|\mathbf{v}\|^2 + \langle \mathbf{a}^1, \mathbf{v} \rangle + a_0 \right],$$

where

$$a_2(T) = \alpha c + \frac{1}{2}T,$$

$$\mathbf{a}^1 = \mathbf{s}^0 - \mathbf{s}^{MSL} - 2\alpha\beta c \mathbf{1}, \quad \mathbf{1} = (1, \dots, 1) \in \mathbf{R}^I,$$

and

$$a_0 = cF(1 + \alpha\beta^2).$$

For each fixed value of  $T$ , the latter expression in  $[\cdot]$ 's is a *quadratic form* [1] in  $\mathbf{v}$ , whose *principal axes* [1] we now exhibit by several changes of variables, thus simplifying minimization. First, on defining

$$\mathbf{u} \stackrel{\text{def}}{=} \sqrt{a_2(T)} \mathbf{v},$$

we obtain

$$a_2(T) \|\mathbf{v}\|^2 + \langle \mathbf{a}^1, \mathbf{v} \rangle = \langle \mathbf{u}, \mathbf{u} \rangle + \left\langle \frac{1}{\sqrt{a_2(T)}} \mathbf{a}^1, \mathbf{u} \right\rangle$$

Next, on defining

$$\mathbf{b}^1(T) \stackrel{\text{def}}{=} \frac{1}{\sqrt{a_2(T)}} \mathbf{a}^1, \quad \mathbf{w} \stackrel{\text{def}}{=} \mathbf{u} + \frac{1}{2} \mathbf{b}^1(T),$$

we obtain

$$\begin{aligned} & a_2(T) \|\mathbf{v}\|^2 + \langle \mathbf{a}^1, \mathbf{v} \rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \left\langle \frac{1}{\sqrt{a_2(T)}} \mathbf{a}^1, \mathbf{u} \right\rangle \\ &= \langle \mathbf{u}, \mathbf{u} \rangle + \langle \mathbf{b}^1(T), \mathbf{u} \rangle \\ &= \langle \mathbf{u} + \mathbf{b}^1(T), \mathbf{u} \rangle \\ &= \left\langle \mathbf{w} + \frac{1}{2} \mathbf{b}^1(T), \mathbf{w} - \frac{1}{2} \mathbf{b}^1(T) \right\rangle \\ &= \|\mathbf{w}\|^2 - \frac{1}{4} \|\mathbf{b}^1(T)\|^2, \end{aligned}$$

which leads to the problem

$$\min_{\mathbf{w}, T} \rightarrow \tilde{\Phi} \stackrel{\text{def}}{=} T \left[ \|\mathbf{w}\|^2 - \frac{1}{4} \|\mathbf{b}^1(T)\|^2 + a_0 \right], \quad (17)$$

subject to the constraint

$$g \stackrel{\text{def}}{=} s_1^0 + T \left( \frac{w_1 - \frac{1}{2} b_1^1(T)}{\sqrt{a_2(T)}} \right) + L_1 = 0 \quad (18)$$

Using a Lagrange multiplier [1]  $\lambda$ , we arrive at the problem of minimizing the Lagrangian

$$\Lambda \stackrel{\text{def}}{=} \tilde{\Phi} - \lambda g$$

by solving the algebraic system

$$\left. \begin{aligned} 0 &= \frac{\partial}{\partial w_i} \Lambda & i &= 1, \dots, F, \\ 0 &= \frac{\partial}{\partial T} \Lambda \\ 0 &= g \end{aligned} \right\} \quad (19)$$

for  $\mathbf{w}, T, \lambda$ .

For  $i = 2, \dots, F$ , one obtains

$$0 = \frac{\partial}{\partial w_i} \Lambda = 2T w_i,$$

whence  $w_i = 0$ .

Thus, regardless of the dimension  $I$ , the above minimization problem reduces to the static 2-dimensional minimization problem

$$\min_{w_1, T} \rightarrow T \left[ w_1^2 - \frac{1}{4} \|\mathbf{b}^1(T)\|^2 + a_0 \right], \quad (20)$$

subject to the constraint (18). This reduction gives the exact coefficients of the following polynomial in the variable  $z = \sqrt{a_2(T)}$ :

$$\left. \begin{aligned} & - z^8 (16a_0) \\ & + z^7 (4L_1 + 4s_1^0 + 4a_1^1) \\ & + z^6 (32a_0 \alpha c) \\ & + z^5 (-4\alpha c L_1 - 4\alpha c s_1^0) \\ & + z^4 (8\alpha c I^2 - 16a_0 \alpha^2 c^2) \\ & + z^3 (8\alpha^2 c^2 L_1 + 8\alpha^2 c^2 s_1^0 - 12a_1^1 \alpha^2 c^2) \\ & + z (8a_1^1 \alpha^3 c^3) \\ & - 16\alpha^2 c^2 z^2 I^2 + 8\alpha^3 c^3 I^2 \end{aligned} \right\} \quad (21)$$

The latter polynomial, obtained using the software [15], is readily seen to have degree 8 independently of  $I$ . One of the roots of the polynomial corresponds to the solution of (20), (18). This gives a computational solution procedure that has the desired features discussed in the latter portion of the section *Introduction*. This knowledge enables a calculation of the solution with arbitrary accuracy for the general  $I$ -aircraft problem P1 in time  $O(I)$  if the intended exit sequence for P1 is pre-determined, and in time  $O(I^2)$  otherwise.



### Obtaining delays and time advances from a solution

Once a solution  $\mathbf{v}(t)$  covering all three flight phases is found, the quantities  $\delta_i$  appearing in (8) are found as follows. For aircraft  $i$ , let  $T_i^1$  be the earliest time when

$$s_i = -L_i,$$

take

$$\delta_i = T_i^1 - STA_i \quad (22)$$

### Numerical experiments for problem P1

The correctness of the results obtained in the previous section was tested by solving several instances of problem P1 using the following three methods.

1. Solving problem (20), (18) by finding numerically a suitable root of (21) and, from it, the corresponding speed advisory  $\mathbf{v}^*$ , using the *wxMaxima* software [15].
2. Solving problem (20), (18) numerically using the `fminsearch` function in the MATLAB software [16].
3. Solving the original,  $I$ -dimensional, optimal control problem P1, given by (7), (8), (9), (10), (12), (13), using the OCP software [14] and *without assuming the constancy of optimal control strategies asserted in Remark 3*.

The obtained values from all three approaches are shown in Tables 1 and 2. For each value of  $I$  (number of aircraft), the value of the coefficient  $c$  (see (12)) was varied over the values

$$c = 10^n, \quad n = -2, -1.8, -1.6, \dots, 6$$

Plots of dependence of the solution and excess separation on the value of  $c$  are shown for 2 and 3 aircraft in Figures 4 and 5, respectively. Figure 6 shows MATLAB solution plots for a case of 30 aircraft.

### The fuel burn curve

Throughout this section, the functional form (11) of instantaneous fuel burn will be used with parameter values

$$\alpha = 5 \times 10^{-5}, \quad \beta = 4.5 \times 10^2,$$

chosen to mimic the data shown in reference [13, figure 1].

### A case of 2 aircraft

The numerical results in Table 1 were obtained assuming that  $i = 1$  exits the delay-time advance phase before the other flight does, and assuming the following parameter values:

$$\left. \begin{aligned} L_1 &= 20.0 \text{ nmi}, \quad L_2 = 30.0 \text{ nmi} \\ \rho &= 5 \text{ nmi} = \text{min. separation required for } i = 2 \text{ trailing } i = 1 \\ \mathbf{s}^{MSL} &= (-L_1, -L_2 - \rho) \\ \mathbf{s}^0 &= (-L_1 - \beta, -L_2 - \beta) \\ \underline{v}_i &= 250 \text{ kts}, \quad \bar{v}_i = 550 \text{ kts} \quad \text{for all } i \end{aligned} \right\} \quad (23)$$

Solution values are shown in Table 1; solution plots, in Figure 4.

### A case of 3 aircraft

The numerical results in Table 2 were obtained assuming that  $i = 1$  exits the delay-time advance phase before the other flights do, and assuming the following parameter values:

$$\left. \begin{aligned} L_1 &= 20.0 \text{ nmi}, \quad L_2 = 30.0 \text{ nmi}, \\ L_3 &= 25.0 \text{ nmi} \\ \rho &= 5 \text{ nmi} = \text{min. separation required for all aircraft pairs} \\ \mathbf{s}^{MSL} &= (-L_1, -L_2 - \rho, -L_3 - 2\rho) \\ \mathbf{s}^0 &= (-L_1 - \beta, -L_2 - \beta, -L_3 - \beta) \\ \underline{v}_i &= 250 \text{ kts}, \quad \bar{v}_i = 550 \text{ kts} \quad \text{for all } i \end{aligned} \right\} \quad (24)$$

**Table 1: Solutions for instances (23) of P1.**

$c$	Method	$\mathbf{v}^*$ (kts)	$T$ (hr)
1.e-2	w(xMaxima)	(4.77e+02, 4.71e+02)	9.44e-01
	M(ATLAB)	(4.74e+02, 4.69e+02)	9.49e-01
	O(CP)	(4.74e+02, 4.69e+02)	9.49e-01
1.e-1	w	(4.74e+02, 4.69e+02)	9.49e-01
	M	(4.74e+02, 4.69e+02)	9.49e-01
	O	(4.74e+02, 4.69e+02)	9.49e-01
1.e+0	w	(4.74e+02, 4.69e+02)	9.49e-01
	M	(4.74e+02, 4.69e+02)	9.49e-01
	O	(4.74e+02, 4.69e+02)	9.49e-01
1.e+1	w	(4.74e+02, 4.69e+02)	9.49e-01
	M	(4.74e+02, 4.69e+02)	9.49e-01
	O	(4.74e+02, 4.69e+02)	9.49e-01
1.e+2	w	(4.74e+02, 4.69e+02)	9.49e-01
	M	(4.74e+02, 4.69e+02)	9.49e-01
	O	(4.74e+02, 4.69e+02)	9.49e-01
1.e+3	w	(4.75e+02, 4.68e+02)	9.47e-01
	M	(4.75e+02, 4.68e+02)	9.47e-01
	O	(4.75e+02, 4.68e+02)	9.47e-01
1.e+4	w	(4.81e+02, 4.62e+02)	9.36e-01
	M	(4.81e+02, 4.62e+02)	9.36e-01
	O	(4.81e+02, 4.62e+02)	9.36e-01
1.e+5	w	(4.90e+02, 4.53e+02)	9.19e-01
	M	(4.90e+02, 4.53e+02)	9.19e-01
	O	(4.90e+02, 4.53e+02)	9.19e-01

Solution values are shown in Table 2; solution plots, following parameter values: in Figure 5.

### ***A case of 30 aircraft***

The timeframe in which the 2- and 3-aircraft instances of P1 were solved, above, is on the order of 1 hour. Within this timeframe, saturation rates of air traffic arriving in a terminal can well exceed 60 aircraft per hour, while minimal rates are typically far in excess of 3 aircraft. Thus, a case of 30 aircraft corresponds to a rate of arrivals which is, on the one hand, below saturation and, on the other hand, more realistic than 3 aircraft. It is at this rate that the advantages of time advance are expected to be most manifest.

The numerical results for this case were obtained assuming that  $i = 1$  exits the delay-time advance phase before the other flights do, and assuming the

$L_i = 20.0(1 + \kappa_i)$  nmi, where  $\kappa_i$  is a random variable [1] distributed uniformly [1] in  $[-0.3, 0.3]$

$\rho = 5$  nmi = min. separation required for all aircraft pairs

$\mathbf{s}^{MSL} = (s_i^{MSL})_i, s_i^{MSL} = -L_i - (i - 1)\rho$

$\mathbf{s}^0 = (-L_i - \beta)_i$  for all  $i$

$\underline{v}_i = 250$  kts,  $\bar{v}_i = 570$  kts for all  $i$

The speed advisories were computed using MATLAB [16] without imposing the speed range constraints. For values of  $c$  above  $10^4$ , the computed speed advisory  $u_i^*$  for aircraft  $i = 1$  exceeded

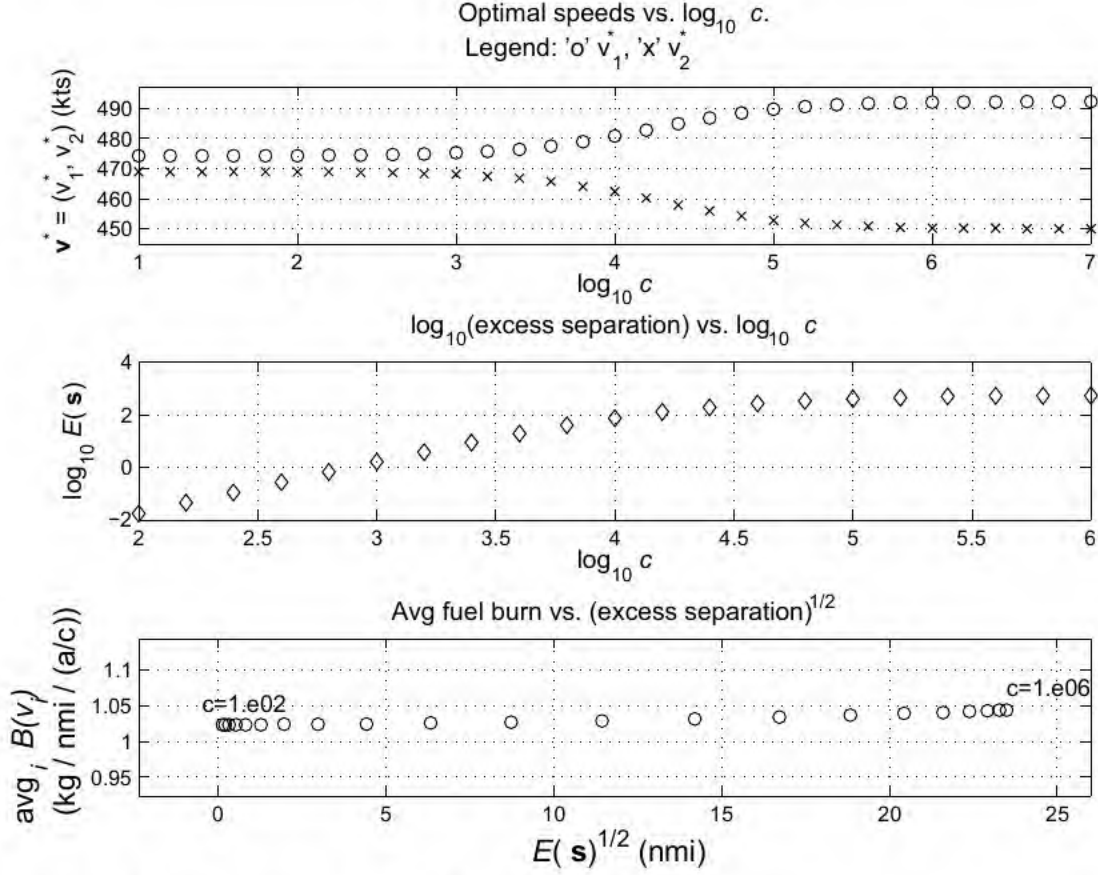


Figure 4: Solutions for the 2 a/c case of P1.

the value  $\bar{v}_i = 570$  kts. This behavior may be an indication that the functional form  $B_{\tau(i)}(u_i)$  of fuel burn used above does not include the highly nonlinear effects arising as a subsonic aircraft approaches the speed of sound, and also that ascribing excessively large priority  $c$  to fuel savings can yield operationally infeasible results. This issue will be addressed as the model and algorithm are further developed to accommodate a higher level of operational realism. For  $c \leq 10^4$ , the computed speed advisories do meet the speed range constraints, the values of  $E(s)$  are below 5 nmi, and fuel burn exhibits a negative correlation with excess separation (see Figure 6). Consequently, fuel savings are negatively correlated with throughput, as expected.

## Discussion

### *Summary: the results and their operational implications*

The optimal control formulated above (problem P1) describes aircraft traffic in its Delay - Time Advance phase of traffic arrival. The cost functional includes two terms, collective for the entire traffic: fuel burn (12) and excess separation (13). These two terms are in competition for priority; e.g., in order to reduce excess separation (thereby increasing throughput), some aircraft must speed up, incurring higher fuel burn. The cost functional thus represents the tradeoff between the pursuits of minimizing the two costs. The relative weight ascribed to fuel burn over excess separation is represented by the coefficient  $c$  appearing in (12).

Similar optimal control models are formulated

**Table 2: Solutions for instances (24) of P1.**

$c$	Method	$\mathbf{v}^*$ (kts)	$T$ (hr)
1.e-2	w(xMaxima)	(4.77e+02, 4.72e+02, 4.66e+02)	9.43e-01
	M(ATLAB)	(4.77e+02, 4.72e+02, 4.66e+02)	9.43e-01
	O(CP)	(4.77e+02, 4.72e+02, 4.66e+02)	9.43e-01
1.e-1	w	(4.77e+02, 4.72e+02, 4.66e+02)	9.43e-01
	M	(4.77e+02, 4.72e+02, 4.66e+02)	9.43e-01
	O	(4.77e+02, 4.72e+02, 4.66e+02)	9.43e-01
1.e+0	w	(4.77e+02, 4.72e+02, 4.66e+02)	9.43e-01
	M	(4.77e+02, 4.72e+02, 4.66e+02)	9.43e-01
	O	(4.77e+02, 4.72e+02, 4.66e+02)	9.43e-01
1.e+1	w	(4.77e+02, 4.72e+02, 4.66e+02)	9.43e-01
	M	(4.77e+02, 4.72e+02, 4.66e+02)	9.43e-01
	O	(4.77e+02, 4.72e+02, 4.66e+02)	9.43e-01
1.e+2	w	(4.77e+02, 4.72e+02, 4.66e+02)	9.43e-01
	M	(4.77e+02, 4.72e+02, 4.66e+02)	9.43e-01
	O	(4.77e+02, 4.72e+02, 4.66e+02)	9.43e-01
1.e+3	w	(4.78e+02, 4.71e+02, 4.66e+02)	9.41e-01
	M	(4.78e+02, 4.71e+02, 4.66e+02)	9.41e-01
	O	(4.78e+02, 4.71e+02, 4.66e+02)	9.41e-01
1.e+4	w	(4.87e+02, 4.65e+02, 4.63e+02)	9.24e-01
	M	(4.87e+02, 4.65e+02, 4.63e+02)	9.24e-01
	O	(4.87e+02, 4.65e+02, 4.63e+02)	9.24e-01
1.e+5	w	(5.06e+02, 4.54e+02, 4.54e+02)	8.90e-01
	M	(5.06e+02, 4.54e+02, 4.54e+02)	8.90e-01
	O	(5.06e+02, 4.54e+02, 4.54e+02)	8.90e-01

(see appendices I and II) to describe the two subsequent phases of each arrival flight. The solution in each model is a *collective speed*  $\mathbf{v}(t)$ , which is generally a time-dependent vector whose  $i$ -th component is the speed advised for flight  $i$  along its intended path. (As is shown above,  $\mathbf{v}(t)$  turns out constant for the first, Delay - Time Advance, phase of arrival.) Once the solutions  $\mathbf{v}(t)$  for all three flight phases are found, they completely determine the optimal (in the sense of the cost functionals) amount of delay or time advance for each aircraft. This determination is given by formula (22).

The solutions obtained for the Delay - Time Advance phase (problem P1) and verified using three independent computational methods (tables 1 and 2) are a step toward automation of scheduling that takes account of fuel burn and its interplay with excess separation. The latter quantity, in turn, af-

fects airport efficiency. The plots shown in Figures 4 and 5 indicate that, *i*) as expected, with the higher “relative importance” (value of  $c$ ) ascribed to fuel burn, excess separation increases toward, and levels out at, a maximal value, and *ii*) the speed advised for the lead aircraft increases. The plot in Figure 6 confirms the intuitive expectation that increasing the priority of fuel savings results in higher excess separation, hence in lower throughput.

Another potential benefit of such quickly and accurately computable solutions is that, with equipage to communicate the parameters  $\alpha, \beta, L_i, \mathbf{s}^0$  to the crew of each flight  $i$  and to carry out simple calculations, each flight crew is in a position to determine its optimal speed advisory that meets the separation constraints between this flight and all the others. The contributions of this paper, consequently, are a step toward *self-separation* [12].



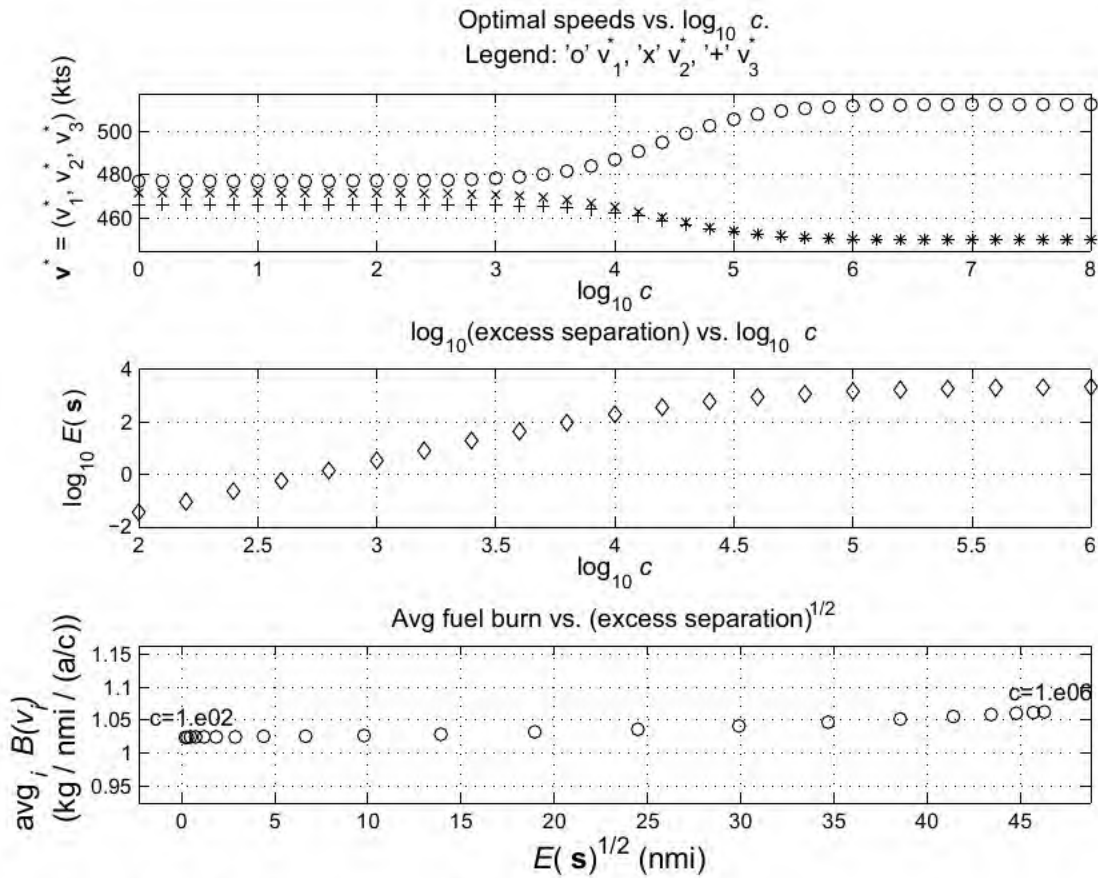


Figure 5: Solutions for the 3 a/c case of P1.

### Directions for future research

The following research directions seem sensible for next steps toward operational realism and development of automation tools fit for use in the field. Introducing uncertainty terms into the control laws describing the traffic can help capture the behavior of traffic in the presence of wind and human factors in control execution. A solution the 3-phase control problem with P1, P2, and P3 as phases would enable formula (22) to be used for calculating the resulting delays and time advances. Inclusion of inertia in the control model can indicate how the cost function may need to be modified in order to capture the resources of importance (e.g., fuel) and to yield speed advisories that, like the ones found above, are piecewise constant (Remark 1). Testing the computational procedures using high-fidelity ATM operations and live traffic.

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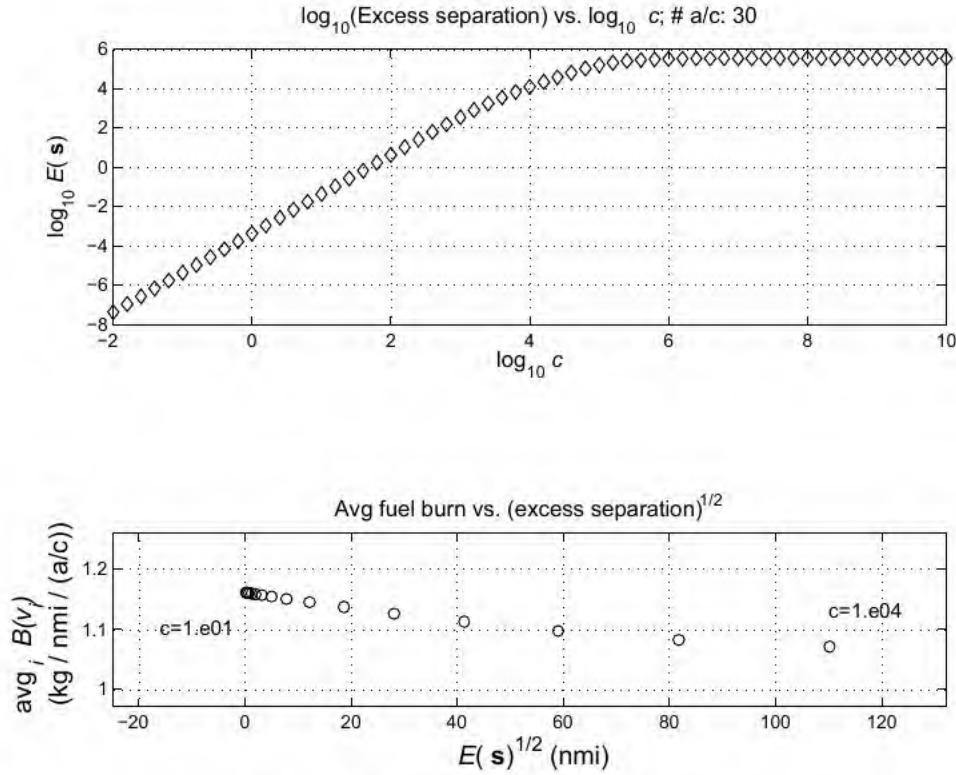


Figure 6: Solution plots for the 30 a/c case of P1.

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## Appendix I: The optimal control problem for the Mixed Phase

This problem, denoted P2, arises once the system reaches a state  $\mathbf{s}^1$  in the target set (9) of problem P1. Given the state  $\mathbf{s}^1$ , which will serve as the initial state for problem P2, let the set  $\mathcal{A}_1$  consist of those indices  $i$  in  $\mathcal{A}$  for which the aircraft still has the option of exercising delay or time advance, i.e.

$$s_i^1 < -L_i$$

Problem P2, while governed by the same control law as P1, allows aircraft  $i$  a narrower speed range

$$[\underline{v}_i, \bar{v}_i]$$

The target set for P2 is the set of all states  $\mathbf{s}$  for which

$$s_i = -L_i \quad \text{for at least one index } i \text{ in } \mathcal{A}_1$$

The running cost includes a term reflecting fuel burn only for those aircraft in  $\mathcal{A}_1$ , but the exit cost is the same as in P1. The resulting Pontryagin function is

$$H = -c \sum_{f \in \mathcal{A}_1} B_{\tau(i)}(v_i) + \langle \mathbf{v}, \boldsymbol{\sigma} \rangle$$

## Appendix II: The optimal control problem for the Separation Minimization Phase

Once each  $s_i$  exceeds  $-L_i$ , none of the aircraft pursues the objective of minimizing fuel burn and is concerned solely with minimizing separation. The control, the speed ranges, and the exit cost are as in P2. The running cost is now zero. The target set is a state  $\mathbf{s}^3$  in which

$$-L_i < s_i < -l_i$$

The Pontryagin function is

$$H = \langle \mathbf{v}, \boldsymbol{\sigma} \rangle$$

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